

States and amplitudes for finite regions in a two-dimensional Euclidean quantum field theory

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We quantize the Helmholtz equation (plus perturbative interactions) in two dimensions to illustrate a manifestly local description of quantum field theory. Using the general boundary formulation we describe the quantum dynamics both in a traditional time evolution setting as well as in a setting referring to finite disk (or annulus) shaped regions of spacetime. We demonstrate that both descriptions are equivalent when they should be.

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I. INTRODUCTION

The general boundary formulation of quantum theory (GBF) [1, 2, 3] offers a new way to study the quantum theory of fields. A main feature of this approach is the possibility to associate Hilbert spaces of states with arbitrary hypersurfaces of spacetime. All the information about the physical processes taking place within a spacetime region is encoded in the amplitude associated with such a region and states on its boundary hypersurface. A key aspect of this approach is the absence of the requirement of a special type of spacetime hypersurfaces for the construction of the quantum theory. The GBF should be implementable for spacetime regions of *arbitrary* form. This peculiar characteristic of the GBF contrasts dramatically with the conventional formulation of quantum field theory where a special class of hypersurfaces is singled out, namely flat spacelike hypersurfaces defined by a constant value of Minkowskian time. Indeed state spaces are defined on such equal-time surfaces and transition amplitudes are defined between two such surfaces. This structure of conventional quantum field theory is the reflection of the unique role played by time in quantum theory as the parameter labeling the evolution.

The first extension of the standard formulation of quantum field theory relevant in the present context has been introduced by Tomonaga and Schwinger in the late 40s [4, 5]. In their works they described the evolution of fields quantized on arbitrary spacelike hypersurfaces through the so called Tomonaga-Schwinger equation, which generalizes the functional Schrödinger equation. Other generalized quantization prescriptions have also been discussed in the literature, among which we recall the light-front dynamics [6], where the surface of quantization is the plane $t + z = \text{const.}$, the covariant formulation of the light-front dynamics [7], in which the wave functions are defined on the plane characterized by the equation $v_\mu x^\mu = 0$ where v is an arbitrary light-like four vector, and the quantization on the Lorentz invariant spacetime hyperboloid $x_\mu x^\mu = \text{const.} > 0$ [8], also known as point-form dynamics¹. The different characteristics of these approaches depend on the properties of the quantization surface, whose exact form is therefore of paramount importance.

The GBF is compatible with all the above mentioned descriptions of quantum field theory. Moreover, it should be possible to view them as special cases. In the GBF one should be able to describe the dynamics of quantized fields in spacetime regions whose geometry is different and even incompatible with the mathematical structures on which the other mentioned approaches are based. To be more precise, in the different formalisms of QFT mentioned above transition amplitudes are defined for evolution processes that involve an infinite spacetime region bounded by two disconnected spacelike (or lightlike) hypersurfaces. Such a geometry is dictated on the one hand by certain quantization prescriptions (given by canonical commutation rules to be imposed on the quantization surface). On the other hand it is imposed by the standard picture of dynamics understood as the evolution from an initial state (defined on an initial spacelike hypersurface) to a final one (defined on a final spacelike hypersurface) and the associated probability interpretation, limited to *transition* amplitudes. But if we are interested in a dynamical process taking place in a spacetime region naturally bounded by, say, one connected hypersurface containing timelike parts, the use of the mentioned forms of QFT may become awkward or even impossible. (As an extreme case think of a stationary

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¹ These different forms to describe dynamics have been advocated by Dirac in [9] at the level of classical relativistic dynamics.

black hole spacetime.) In contrast, the GBF can handle such a case at least without conceptual difficulty. The *technical* problems however, may be considerable in general.

The conjecture that standard QFT admits an extension in the sense of the GBF has been addressed in a series of papers: In [10] it was shown (in the context of free scalar QFT) that states on certain *timelike* hypersurfaces and amplitudes between them can be consistently defined and interpreted. A first example of a region with a timelike and *connected* boundary was given in [11]. The region in question is a timelike hypercylinder, i.e., a ball in space extended over all of time. Here, due to the connectedness of the boundary, amplitudes cannot be thought of as transition amplitudes between different boundary components. Nevertheless, a consistent probability interpretation was demonstrated. Perturbatively interacting QFT was treated in [12, 13] showing that an interacting asymptotic amplitude can be defined from the large radius limit of the hypercylinder. Moreover, it was demonstrated that this amplitude is equivalent to the usual S-matrix when both can be defined.

Although the connectedness of the boundary is certainly a novelty, the hypercylinder shares with the other geometries considered in the literature so far the property of being non-compact. In the present article we make a more radical departure from standard QFT and consider a *finite* region with connected boundary.² In contrast to the other mentioned articles, we operate here in a setting of *Euclidean* spacetime, as this leads to considerable technical simplifications. We study the quantization of a field theory obeying a Helmholtz equation of motion and its perturbations in two spacetime dimensions (the generalization to higher dimensions presents no conceptual difficulty). Two different geometries are studied. First, we consider the case of parallel “equal-time” hyperplanes with the enclosed region representing “time-evolution”. This corresponds to a traditional setup, describing the quantum theory in terms of states evolving in time. Then, we consider hypersurfaces given by concentric circles in spacetime and the regions enclosed by them, i.e., discs and annuli of different radii. The annuli amplitudes can be thought of as describing the “evolution” of states between different radii. In contrast, the disc amplitudes have no conventional “transition” interpretation. The conceptual meaning of the amplitudes is similar to that in the hypercylinder setting first described in [11].

The plan of this work largely follows that of [13]. We consider in Section II the classical field theory in the two settings mentioned above, and we express the classical solutions of the equation of motion in terms of the boundary configurations in the different settings. In Section III the quantum theory is presented in the Schrödinger representation and the field propagators for the different regions considered are specified. Vacuum and coherent states are introduced in Section IV and V respectively. Amplitudes for the free theory in both geometries are evaluated in Section VI and their relation studied in Section VII, where we construct an isometry between the respective state spaces which makes the amplitudes equal in the interaction picture. In Section VIII we introduce sources to describe perturbative interactions and construct asymptotic amplitudes of the interacting theory for both geometries (one of them being essentially the conventional S-matrix). The case of general interactions is treated in Section IX by means of functional techniques. Finally we present our conclusion in Section X.

Note that, although we work in Euclidean spacetime, our setting is conceptually distinct from considering the Wick-rotation of a Lorentzian theory. Rather, we view the theory here as a *Riemannian* real-time QFT in its own right. In particular, although there are certain formal similarities, what we are doing is essentially different from the technique known as *radial quantization* [8], where one maps a Lorentzian cylinder to a Euclidean plane. In radial quantization Hilbert spaces can also be thought of as associated to circles in a plane, but the latter are interpreted as images of usual equal-time hypersurfaces in a Lorentzian cylinder, mapped to the plane. A consequence of this conceptual difference is, for example, that the Hilbert space associated in the present setting to a circle corresponds to a tensor product of *two* equal-time Hilbert spaces rather than to just one (as would be the case in radial quantization).

II. CLASSICAL THEORY

We consider a real massive scalar field in 2 dimensional Euclidean spacetime which obeys a Helmholtz equation of motion. We will be interested in studying the field in two different kinds of spacetime regions. On the one hand we consider an infinite region bounded by two parallel straight lines. This represents the traditional point of view on quantum field theory of evolution between equal time hypersurfaces. On the other hand we consider a region with the shape of a disk (and also a region with the shape of an annulus). This is more adapted to the idea that we want to describe physics locally.

² Note that finite regions in a GBF context have been considered already in the case of Yang-Mills theory in two dimensions [14]. However, this theory is almost topological and thus differs substantially from realistic QFTs.

Choosing Cartesian coordinates τ, x we consider the action

$$S_{M,0}(\phi) = \frac{1}{2} \int_M d\tau dx ((\partial_\tau \phi)(\partial_\tau \phi) + (\partial_x \phi)(\partial_x \phi) - m^2 \phi^2), \quad (1)$$

for a region M in spacetime. The associated equation of motion is the Helmholtz equation,

$$(\partial_\tau^2 + \partial_x^2 + m^2) \phi = 0. \quad (2)$$

A. Slice regions

Consider the spacetime region $M = [\tau_1, \tau_2] \times \mathbb{R}$, where we may think of $[\tau_1, \tau_2]$ as a “time interval”. Bounded solutions of (2) in the region M can be expanded in Fourier modes in “space” and either in Fourier modes or in exponentials in “time”,

$$\phi(\tau, x) = \int_{-\infty}^{+\infty} \frac{d\nu}{2\pi} (a(\nu)e^{i\omega_\nu \tau} + b(\nu)e^{-i\omega_\nu \tau}) e^{i\nu x}. \quad (3)$$

Here the “time” variable τ belongs to the interval $[\tau_1, \tau_2]$ and we define

$$\omega_\nu := \begin{cases} \sqrt{m^2 - \nu^2} & \text{if } |\nu| \leq m \\ -i\sqrt{\nu^2 - m^2} & \text{if } |\nu| > m. \end{cases} \quad (4)$$

For $|\nu| > m$ the field is a combination of solutions oscillating in space and exponentially increasing or decreasing in time. For $|\nu| \leq m$, the field is oscillating both in space and time. The conditions for the field to be real are $\overline{a(\nu)} = b(-\nu)$ and $\overline{b(\nu)} = a(-\nu)$ if $|\nu| \leq m$. Otherwise they are $\overline{a(\nu)} = a(-\nu)$ and $\overline{b(\nu)} = b(-\nu)$.

It will be useful in the following to work with solutions of the equation of motion with specific boundary conditions. In particular a field ϕ , obeying equation (2), with boundary configurations φ_1 at $\tau = \tau_1$ and φ_2 at $\tau = \tau_2$ can be formally written as³

$$\phi(\tau, x) = \frac{\sin \omega(\tau_2 - \tau)}{\sin \omega(\tau_2 - \tau_1)} \varphi_1(x) + \frac{\sin \omega(\tau - \tau_1)}{\sin \omega(\tau_2 - \tau_1)} \varphi_2(x). \quad (5)$$

The symbol ω denotes the operator whose eigenvalues on a Fourier expansion in the x -direction are the values ω_ν .

B. Disk and annulus regions

We now consider spacetime regions with the shape of a disc and with the shape of an annulus. For this purpose it is convenient to introduce polar coordinates r, ϑ where $\tau = r \sin \vartheta$ and $x = r \cos \vartheta$. The equation of motion (2) in polar coordinates is,

$$\left(\partial_r^2 + \frac{1}{r} \partial_r + \frac{1}{r^2} \partial_\vartheta^2 + m^2 \right) \phi = 0. \quad (6)$$

We expand solutions in terms of Fourier modes around the circle,

$$\phi(r, \vartheta) = \sum_{n=-\infty}^{+\infty} f_n(r) e^{in\vartheta}. \quad (7)$$

Local solutions of (6) can then be written in terms of Bessel functions,

$$f_n(r) = a_n J_n(mr) + b_n Y_n(mr). \quad (8)$$

³ Here and in the following we use the symbol ϕ for field configurations in spacetime regions, while we use the symbol φ for field configurations on hypersurfaces.

J_n and Y_n are the Bessel functions of the first and second kind respectively. The conditions for the field to be real read $\overline{a_n} = (-1)^n a_{-n}$ and $\overline{b_n} = (-1)^n b_{-n}$. Notice that these Bessel functions have different behavior at the origin ($r = 0$),

$$J_0(0) = 1, \quad J_n(0) = 0 \quad (n = \pm 1, \pm 2, \dots), \quad \lim_{r \rightarrow 0} Y_n(r) = -\infty \quad \forall n. \quad (9)$$

Consider now a disk region of radius R around the origin. Due to the divergence of the Bessel functions of the second kind at the origin, only Bessel functions of the first kind are admissible in solutions. Given boundary data $\varphi(\vartheta)$ at radius R we can thus formally reconstruct a solution in the whole disk,

$$\phi(r, \vartheta) = \frac{J_n(mr)}{J_n(mR)} \varphi(\vartheta). \quad (10)$$

The fraction $\frac{J_n(mr)}{J_n(mR)}$ is to be understood as an operator, defined through its eigenvalues on a Fourier decomposition on the circle.

We will be interested also in the annulus region, i.e., the region bounded by two circles around the origin. In the annulus region the classical solution may contain both kinds of Bessel functions because both J_n and Y_n are regular there. Hence the classical solution can be decomposed formally in the form

$$\phi(r, \vartheta) = J_n(mr) \varphi_J(\vartheta) + Y_n(mr) \varphi_Y(\vartheta), \quad (11)$$

where φ_J and φ_Y are real functions on the circle. $J_n(mr)$ and $Y_n(mr)$ are understood as operators, defined through their eigenvalues on a Fourier decomposition on the circle. Denoting with φ and $\hat{\varphi}$ the configurations on the boundaries of the annulus specified by the circles of radii R and \hat{R} , with $\hat{R} > R$, respectively, we express the classical solution (11) in terms of the boundary configurations as we did in (10),

$$\begin{pmatrix} \varphi \\ \hat{\varphi} \end{pmatrix} = \begin{pmatrix} J_n(mR) & Y_n(mR) \\ J_n(m\hat{R}) & Y_n(m\hat{R}) \end{pmatrix} \begin{pmatrix} \varphi_J \\ \varphi_Y \end{pmatrix}. \quad (12)$$

Inverting we arrive at

$$\begin{pmatrix} \varphi_J \\ \varphi_Y \end{pmatrix} = \frac{1}{\delta_n(mR, m\hat{R})} \begin{pmatrix} Y_n(m\hat{R}) & -Y_n(mR) \\ -J_n(m\hat{R}) & J_n(mR) \end{pmatrix} \begin{pmatrix} \varphi \\ \hat{\varphi} \end{pmatrix}, \quad (13)$$

where

$$\delta_n(z, \hat{z}) := J_n(z) Y_n(\hat{z}) - Y_n(z) J_n(\hat{z}). \quad (14)$$

Hence, the classical solution in the annulus region takes the form

$$\phi(r, \vartheta) = \frac{J_n(mr) Y_n(m\hat{R}) - Y_n(mr) J_n(m\hat{R})}{\delta_n(mR, m\hat{R})} \varphi(\vartheta) + \frac{J_n(mR) Y_n(mr) - Y_n(mR) J_n(mr)}{\delta_n(mR, m\hat{R})} \hat{\varphi}(\vartheta). \quad (15)$$

III. QUANTUM THEORY

The passage to the quantum theory is implemented by the Feynman path integral prescription, which is the quantization procedure most suited for the GBF. Moreover, the quantum dynamics of the field is described in the Schrödinger representation, where the quantum states are wave functionals on the space of field configurations. Thus, we associate state spaces \mathcal{H}_Σ of wave functions with certain hypersurfaces Σ in spacetime. Amplitudes $\rho_M : \mathcal{H}_{\partial M} \rightarrow \mathbb{C}$ are associated to certain spacetime regions M . (Here ∂M denotes the boundary of M .) State spaces and amplitudes satisfy a number of consistency conditions, see [2] or [11].

The amplitude associated with a region M and a state ψ is given by

$$\rho_M(\psi) = \int d\varphi \psi(\varphi) Z_M(\varphi), \quad (16)$$

where the integral is extended over all the configurations φ on the boundary of the region M . $Z_M(\varphi)$ is the field propagator, formally defined as

$$Z_M(\varphi) = \int_{\phi|_{\partial M} = \varphi} \mathcal{D}\phi e^{iS_M(\phi)}, \quad (17)$$

where $S_M(\phi)$ is the action of the field in M . In the case of the free theory determined by the free action (1) we can evaluate the associated propagator $Z_{M,0}$ by shifting the integration variable by a classical solution matching the boundary configuration φ in ∂M . Explicitly,

$$Z_{M,0}(\varphi) = \int_{\phi|_{\partial M}=\varphi} \mathcal{D}\phi e^{iS_{M,0}(\phi)} = \int_{\phi|_{\partial M}=0} \mathcal{D}\phi e^{iS_{M,0}(\phi_{cl}+\phi)} = N_{M,0} e^{iS_{M,0}(\phi_{cl})}, \quad (18)$$

where the normalization factor is formally given by

$$N_{M,0} = \int_{\phi|_{\partial M}=0} \mathcal{D}\phi e^{iS_{M,0}(\phi)}. \quad (19)$$

In order to identify states and amplitudes explicitly, it is convenient to proceed in the following order: 1. Work out amplitudes and verify their gluing properties. 2. Identify the vacuum state for each relevant hypersurface. 3. Identify particle states or other relevant states. In the following we will carry this out both for the slice regions of Section II A and for the disc or annulus type regions of Section II B.

A. Propagator in the slice region

We evaluate the propagator associated with the slice spacetime region $M = [\tau_1, \tau_2] \times \mathbb{R}$. (In the formulas below we indicate such a region with the subscript $[\tau_1, \tau_2]$.) The boundary ∂M is the disjoint union of the lines $\tau = \tau_1$ and $\tau = \tau_2$. The boundary state $\psi(\varphi)$ results to be the product of the wave function $\psi_1(\varphi_1)$, describing the state of the system at "time" τ_1 , with the wave function $\psi_2(\varphi_2)$, describing the state of the system at "time" τ_2 . By (18) we may use the classical solution (5) of the equation of motion that interpolates between φ_1 at τ_1 and φ_2 at τ_2 in order to express the field propagator of the free theory. The result is,

$$Z_{[\tau_1, \tau_2],0}(\varphi_1, \varphi_2) = N_{[\tau_1, \tau_2],0} \exp \left(\frac{i}{2} \int dx (\varphi_1 \ \varphi_2) W_{[\tau_1, \tau_2]} \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} \right), \quad (20)$$

where the normalization factor is formally given by

$$N_{[\tau_1, \tau_2],0} = \int_{\substack{\phi|_{\tau_1}=0 \\ \phi|_{\tau_2}=0}} \mathcal{D}\phi e^{iS_{[\tau_1, \tau_2],0}(\phi)}, \quad (21)$$

and $W_{[\tau_1, \tau_2]}$ is the operator valued 2×2 matrix

$$W_{[\tau_1, \tau_2]} = \frac{\omega}{\sin \omega(\tau_2 - \tau_1)} \begin{pmatrix} \cos \omega(\tau_2 - \tau_1) & -1 \\ -1 & \cos \omega(\tau_2 - \tau_1) \end{pmatrix}. \quad (22)$$

The relevant gluing property in this context is the composition of two "temporally" consecutive slice regions. We do not write down this calculation explicitly here.

B. Propagator in the disk and annulus regions

As already mentioned, we are interested in two types of regions here: The disk region of radius R , indicated with label R , and the annulus region, bounded by two circles of radii R and \hat{R} (we assume $R < \hat{R}$), indicated with label $[R, \hat{R}]$. For the disk region the field propagator is evaluated using (18) with the classical solution (10),

$$Z_{R,0}(\varphi) = N_{R,0} \exp \left(\frac{i}{2} \int d\vartheta mR \varphi(\vartheta) \frac{J'_n(mR)}{J_n(mR)} \varphi(\vartheta) \right). \quad (23)$$

For the annulus region we use the classical solution (11) and obtain for the propagator the expression

$$Z_{[R, \hat{R}],0}(\varphi, \hat{\varphi}) = N_{[R, \hat{R}],0} \exp \left(\frac{i}{2} \int d\vartheta (\varphi \ \hat{\varphi}) W_{[R, \hat{R}]} \begin{pmatrix} \varphi \\ \hat{\varphi} \end{pmatrix} \right), \quad (24)$$

with

$$W_{[R, \hat{R}]} = \frac{m}{\delta_n(mR, m\hat{R})} \begin{pmatrix} R\sigma_n(m\hat{R}, mR) & -\frac{2}{\pi m} \\ -\frac{2}{\pi m} & \hat{R}\sigma_n(mR, m\hat{R}) \end{pmatrix}. \quad (25)$$

The function δ_n has been defined in (14). The function σ_n is to be understood as the operator defined as

$$\sigma_n(\hat{z}, z) := J_n(\hat{z}) Y'_n(z) - J'_n(z) Y_n(\hat{z}). \quad (26)$$

It can be shown that the propagators (23) and (24) satisfy the following composition rules,

$$Z_{\hat{R},0}(\hat{\varphi}) = \int \mathcal{D}\varphi Z_{R,0}(\varphi) Z_{[R,\hat{R}],0}(\varphi, \hat{\varphi}), \quad (27)$$

and, for $R_1 < R_2 < R_3$,

$$Z_{[R_1,R_3],0}(\varphi_1, \varphi_3) = \int \mathcal{D}\varphi_2 Z_{[R_1,R_2],0}(\varphi_1, \varphi_2) Z_{[R_2,R_3],0}(\varphi_2, \varphi_3). \quad (28)$$

These relations prove the consistency of the definitions (23) and (24).

IV. VACUUM STATE

The vacuum state has to satisfy the vacuum axioms [2], see also [11]. This implies in particular that it is invariant under “evolution”. In the case of the vacuum on an “equal time” line this means “time evolution” along a slice region. In the case of the vacuum on the circle this means (generalized) invariance under radial evolution along an annulus region.

As in [10, 11] we make for the vacuum wave function on a hypersurface Σ the Gaussian ansatz

$$\psi_{\Sigma,0}(\varphi) = C \exp \left(-\frac{1}{2} \int_{\Sigma} dx \varphi(x) (A\varphi)(x) \right), \quad (29)$$

where C is a normalization factor and A an unknown operator.

A. Constant τ line

In the first case we want to determine the vacuum on hypersurfaces that are lines of constant τ . To determine the operator A we consider the free evolution from τ_1 to τ_2 encoded in the propagator (20). The invariance of the vacuum state can be written in the following form

$$\psi_0(\varphi_2) = \int \mathcal{D}\varphi_1 \psi_0(\varphi_1) Z_{[\tau_1,\tau_2],0}(\varphi_1, \varphi_2). \quad (30)$$

This equation implies for the operator A to satisfy $A^2 = \omega^2$. We select the solution $A = \omega$ so that A be bounded from below. Hence, the vacuum state can be written in momentum space as

$$\psi_0(\varphi) = C \exp \left(-\frac{1}{2} \int_{-\infty}^{+\infty} \frac{d\nu}{2\pi} \varphi(\nu) \omega_{\nu} \varphi(-\nu) \right). \quad (31)$$

The normalization factor C is then fixed (up to a phase),

$$|C|^{-2} = \int \mathcal{D}\varphi \exp \left(-\frac{1}{2} \int_{-m}^m \frac{d\nu}{2\pi} \varphi(\nu) 2\omega_{\nu} \varphi(-\nu) \right). \quad (32)$$

B. Circle

For the vacuum state defined on the circle of radius R , the operator $A \equiv A_R$ denotes a family of operators indexed by the radius R . Demanding generalized invariance of the vacuum state under evolution, namely

$$\psi_{R,0}(\varphi) = \int \mathcal{D}\hat{\varphi} \psi_{\hat{R},0}(\hat{\varphi}) Z_{[R,\hat{R}],0}(\varphi, \hat{\varphi}) \quad (33)$$

leads to an equation for the operator A_R ,

$$\left(\text{i}m\sigma_n(m\hat{R}, mR) + \delta_n(mR, m\hat{R})A_R \right) \left(-\text{i}m\sigma_n(mR, m\hat{R}) + \delta_n(mR, m\hat{R})A_{\hat{R}} \right) = \frac{4}{\pi^2 R \hat{R}}. \quad (34)$$

The solutions of the above equation are of the form

$$A_R = \text{i}m \frac{\mathcal{C}'_n}{\mathcal{C}_n}, \quad (35)$$

where \mathcal{C}_n can be one of the following cylindrical functions: Bessel functions of the first and second kind, J_n and Y_n , or Hankel functions of the first and second kind, defined respectively as $H_n = J_n + \text{i}Y_n$ and $\overline{H}_n = J_n - \text{i}Y_n$. If we require that the argument of the exponential in the vacuum state be bounded from below, we must select the Hankel function of the second kind as the cylindrical function in (35). The vacuum state can then be written as

$$\psi_{R,0}(\vartheta) = C_R \exp \left(-\frac{\text{i}}{2} \int d\vartheta \varphi(\vartheta) \left(mR \frac{\overline{H}'_n(mR)}{\overline{H}_n(mR)} \varphi \right) (\vartheta) \right). \quad (36)$$

The normalization factor is given (up to a phase) by the equation

$$|C_R|^{-2} = \int \mathcal{D}\varphi \exp \left(-\frac{2}{\pi} \int d\vartheta \varphi(\vartheta) \left(\frac{1}{|H_n(mR)|^2} \varphi \right) (\vartheta) \right) \quad (37)$$

C. Comparison between the vacuum states

In order to compare the vacuum states in the two settings, we express the operators that define these two vacua, namely ω and A_R obtained in the preceding sections, in an appropriate asymptotic region. In particular we consider a small region near the positive τ axis at large radius R . In polar coordinates this means we take $\vartheta \approx \pi/2$ at $r = R$. Hence we have $\partial_x \approx \frac{1}{R} \partial_\vartheta$. Then, the operator ω in this asymptotic region now reads

$$\omega \approx \sqrt{\frac{\partial_\vartheta^2}{R^2} + m^2}. \quad (38)$$

On the Fourier expansion (7) this yields the eigenvalues $\sqrt{-\frac{n^2}{R^2} + m^2}$. Fixing n we take the limit $R \rightarrow \infty$ to obtain $\omega \rightarrow m$ in this sense. With the asymptotic expansion of the Bessel functions J_n and Y_n (see [15] for example) it is easy to evaluate the expression of A_R for large radius: $A_R \rightarrow m$. Hence, in this asymptotic sense, $\omega \approx A_R$. With another choice in (35) this would not be true. The freedom we found in selecting the vacuum on a constant τ line is indeed linked in this way to the freedom in selecting the vacuum on the circle.

V. COHERENT STATES

In [12, 13] coherent states have been an essential tool for the computation of asymptotic amplitudes. This is equally the case in the present paper. To define coherent states we use an approach parallel to the one in [13].

A. Constant τ line

We define a coherent state at constant τ in the Fock representation as

$$|\psi_\eta\rangle = C_\eta \exp \left(\int_{-m}^m \frac{d\nu}{2\pi} \frac{1}{2\omega_\nu} \eta(\nu) a^\dagger(\nu) \right) |0\rangle, \quad (39)$$

where $a^\dagger(\nu)$ is the creation operator associated with the mode ν of the field, $|0\rangle$ represents the vacuum state and $\eta(\nu)$ is a complex function on the interval $[-m, m]$. Note that the restriction of ν to this interval is analogous to the restriction in [13] of the coherent states on the hypercylinder to depend only on physical configurations. The meaning

of “physical configurations” in the present context is that those are the modes that behave well (i.e., are bounded) for large τ . The normalization factor is

$$C_\eta = \exp \left(-\frac{1}{2} \int_{-m}^m \frac{d\nu}{2\pi} \frac{1}{2\omega_\nu} |\eta(\nu)|^2 \right). \quad (40)$$

In the Schrödinger representation the coherent state reads

$$\psi_\eta(\varphi) = K_\eta \exp \left(\int_{-m}^m \frac{d\nu}{2\pi} \eta(\nu) \varphi(\nu) \right) \psi_0(\varphi), \quad (41)$$

where

$$K_\eta = \exp \left(-\frac{1}{2} \int_{-m}^m \frac{d\nu}{2\pi} \frac{1}{2\omega_\nu} (\eta(\nu)\eta(-\nu) + |\eta(\nu)|^2) \right). \quad (42)$$

The characteristic property of coherent states is to remain coherent under the action of the free propagator: Coherent states evolve to coherent states,

$$\psi_{\eta_2}(\varphi_2) = \int \mathcal{D}\varphi_1 \psi_{\eta_1}(\varphi_1) Z_{[\tau_1, \tau_2], 0}(\varphi_1, \varphi_2). \quad (43)$$

This equation yields the following relation for the complex functions η_1 , defined on the surface $\tau = \tau_1$ and η_2 , defined on the surface $\tau = \tau_2$,

$$\eta_2(\nu) = \eta_1(\nu) e^{-i\omega_\nu(\tau_2 - \tau_1)}. \quad (44)$$

In the interaction picture a coherent state can thus be defined as

$$\psi_{\tau, \eta}(\varphi) = K_{\tau, \eta} \exp \left(\int_{-m}^m \frac{d\nu}{2\pi} \eta(\nu) e^{-i\omega_\nu \tau} \varphi(\nu) \right) \psi_0(\varphi), \quad (45)$$

where the normalization factor $K_{\tau, \eta}$ is now time dependent,

$$K_{\tau, \eta} = \exp \left(-\frac{1}{2} \int_{-m}^m \frac{d\nu}{2\pi} \frac{1}{2\omega_\nu} (e^{-2i\omega_\nu \tau} \eta(\nu)\eta(-\nu) + |\eta(\nu)|^2) \right). \quad (46)$$

It will be useful to expand the coherent state (39) in terms of multiparticle states,

$$|\psi_\eta\rangle = C_\eta \sum_{n=0}^{\infty} \frac{1}{n!} \int_{-m}^m \frac{d\nu_1}{2\pi 2\omega_{\nu_1}} \cdots \frac{d\nu_n}{2\pi 2\omega_{\nu_n}} \eta(\nu_1) \cdots \eta(\nu_n) |\psi_{\nu_1, \dots, \nu_n}\rangle. \quad (47)$$

The inner product between a coherent state defined by the complex function η and a state with n particles of quantum numbers ν_1, \dots, ν_n is,

$$\langle \psi_{\nu_1, \dots, \nu_n} | \psi_\eta \rangle = C_\eta \eta(\nu_1) \cdots \eta(\nu_n). \quad (48)$$

The inner product of two n -particle states is

$$\langle \psi_{\nu_1, \dots, \nu_n} | \psi_{\mu_1, \dots, \mu_n} \rangle = \frac{1}{n!} \sum_{\sigma \in S_n} \prod_{i=1}^n 2\pi 2\omega_{\nu_i} \delta(\nu_i - \mu_{\sigma(i)}). \quad (49)$$

The sum runs over all permutations σ of n elements.

B. Circle

We define the coherent states on the circle of radius R in terms of complex coefficients η_n as

$$\psi_{R, \eta}(\varphi) = K_{R, \eta} \exp \left(\sum_n \eta_n \varphi_n \right) \psi_{R, 0}(\varphi), \quad (50)$$

where the normalization factor is

$$K_{R,\eta} = \exp \left(- \sum_n \frac{|H_n(mR)|^2}{16} (\eta_n \eta_{-n} + |\eta_n|^2) \right). \quad (51)$$

Coherent states remain coherent under free propagation, namely

$$\psi_{R,\eta}(\varphi) = \int \mathcal{D}\hat{\varphi} \psi_{\hat{R},\hat{\eta}}(\hat{\varphi}) Z_{[R,\hat{R}],0}(\varphi, \hat{\varphi}). \quad (52)$$

This equation is satisfied provided that the complex functions η at radius R and $\hat{\eta}$ at radius \hat{R} are related by

$$\eta_n = \hat{\eta}_n \frac{\overline{H}_n(m\hat{R})}{\overline{H}_n(mR)}. \quad (53)$$

Then the interaction picture can be defined with the coefficients $\xi_n = \overline{H}_n(mR)\eta_n$, and a coherent state takes the form

$$\psi_{R,\xi}(\varphi) = K_{R,\xi} \exp \left(\sum_n \frac{\xi_n}{\overline{H}_n(mR)} \varphi_n \right) \psi_{R,0}(\varphi). \quad (54)$$

The normalization factor is then

$$K_{R,\xi} = \exp \left(- \frac{1}{16} \sum_n \left[\frac{H_n(mR)}{\overline{H}_{-n}(mR)} \xi_n \xi_{-n} + |\xi_n|^2 \right] \right). \quad (55)$$

The coherent states satisfy the following completeness relation,

$$D^{-1} \int d\xi d\bar{\xi} |\psi_\xi\rangle \langle \psi_\xi| = I, \quad (56)$$

where I is the identity operator and the constant D has the form

$$D = \int d\xi d\bar{\xi} \exp \left(- \sum_n \frac{|\xi_n|^2}{8} \right). \quad (57)$$

The k -particle expansion of the coherent state determined by the complex function ξ reads

$$|\psi_\xi\rangle = \exp \left(- \sum_n \frac{|\xi_n|^2}{16} \right) \sum_{k=0}^{\infty} \frac{1}{k!} \sum_{n_1} \cdots \sum_{n_k} \xi_{n_1} \cdots \xi_{n_k} |\psi_{n_1, \dots, n_k}\rangle, \quad (58)$$

where $|\psi_{n_1, \dots, n_k}\rangle$ denotes the state with k particles of quantum numbers n_1, \dots, n_k . The inner product between coherent state ψ_ξ and an k -particle state then results to be

$$\langle \psi_{n_1, \dots, n_k} | \psi_\xi \rangle = \exp \left(- \sum_n \frac{|\xi_n|^2}{16} \right) \frac{\xi_{n_1}}{8} \cdots \frac{\xi_{n_k}}{8}. \quad (59)$$

The inner product between two k -particle states is

$$\langle \psi_{n_1, \dots, n_k} | \psi_{n'_1, \dots, n'_k} \rangle = \frac{8^{-k}}{k!} \sum_{\sigma \in S_k} \prod_{i=1}^k \delta_{n_{\sigma(i)}, n'_i}, \quad (60)$$

where the sum is over all the permutations σ of k elements.

VI. AMPLITUDES IN THE FREE THEORY

We compute in the following the amplitudes of coherent states in the theory determined by the free action (1). We use the interaction picture so that amplitudes will be independent of “elapsed time” τ or radius r . In particular, we may think of the amplitudes obtained as asymptotic amplitudes.

A. Slice region

The transition amplitude from the coherent state defined by the complex function η_1 on the hypersurface $\tau = \tau_1$ to the coherent state defined by η_2 on the hypersurface $\tau = \tau_2$ in the interaction picture is given by

$$\rho_{[\tau_1, \tau_2], 0}(\psi_{\tau_1, \eta_1} \otimes \overline{\psi_{\tau_2, \eta_2}}) = \int \mathcal{D}\varphi_1 \mathcal{D}\varphi_2 \psi_{\tau_1, \eta_1}(\varphi_1) \overline{\psi_{\tau_2, \eta_2}(\varphi_2)} Z_{[\tau_1, \tau_2], 0}(\varphi_1, \varphi_2). \quad (61)$$

The above expression reduces to the expression of the inner product between two coherent states defined by the complex functions η_1 and η_2 ,

$$\langle \psi_{\eta_2} | \psi_{\eta_1} \rangle = \exp \left(\int_{-m}^m \frac{d\nu}{(2\pi)2\omega_\nu} \left(\eta_1(\nu) \overline{\eta_2(\nu)} - \frac{1}{2} |\eta_1(\nu)|^2 - \frac{1}{2} |\eta_2(\nu)|^2 \right) \right), \quad (62)$$

and is therefore independent of the initial and final times τ_1 and τ_2 . We can view this as the S-matrix of the free theory, sending $\tau_1 \rightarrow -\infty$ and $\tau_2 \rightarrow +\infty$,

$$\mathcal{S}_0(\psi_{\eta_1} \otimes \overline{\psi_{\eta_2}}) = \lim_{\substack{\tau_1 \rightarrow -\infty \\ \tau_2 \rightarrow +\infty}} \rho_{[\tau_1, \tau_2], 0}(\psi_{\tau_1, \eta_1} \otimes \overline{\psi_{\tau_2, \eta_2}}) = \langle \psi_{\eta_2} | \psi_{\eta_1} \rangle. \quad (63)$$

B. Disk region

The amplitude associated with a coherent state in the interaction picture in the disk region is

$$\begin{aligned} \rho_{R, 0}(\psi_{R, \xi}) &= \int \mathcal{D}\varphi \psi_{R, \xi}(\varphi) Z_{R, 0}(\varphi), \\ &= N_{R, 0} K_{R, \xi} C_R \int \mathcal{D}\varphi \exp \left(\sum_n \left[\frac{\xi_n}{H_n(mR)} \varphi_n - \varphi_n \frac{2}{H_n(mR) J_n(mR)} \varphi_{-n} \right] \right). \end{aligned} \quad (64)$$

We shift the integration variable by the quantity

$$\Delta_n = \frac{J_{-n}(mR)}{4} \xi_{-n}, \quad (65)$$

and we obtain the amplitude

$$\rho_{R, 0}(\psi_{R, \xi}) = \exp \left(\sum_n \frac{1}{16} ((-1)^n \xi_n \xi_{-n} - |\xi_n|^2) \right). \quad (66)$$

Notice that this amplitude is independent of the radius R , as it should be by construction. The limit $R \rightarrow \infty$ is then trivial, and the asymptotic amplitude, \mathcal{S}_0 , is therefore

$$\mathcal{S}_0(\psi_\xi) = \lim_{R \rightarrow \infty} \rho_{R, 0}(\psi_{R, \xi}) = \exp \left(\sum_n \frac{1}{16} ((-1)^n \xi_n \xi_{-n} - |\xi_n|^2) \right). \quad (67)$$

VII. RELATION BETWEEN STATES AND AMPLITUDES IN THE TWO SETTINGS

So far we have kept the description of the two settings (slice regions with line boundaries versus disk/annulus regions with circle boundaries) completely separate. However, the objects (states and amplitudes) that we have constructed in the two settings should be compatible with each other. There is indeed a way to ensure this compatibility. Consider a slice region $[\tau_1, \tau_2] \times \mathbb{R}$ where we cut out a disk of radius R around the origin. (Assume $\tau_1 < -R$ and $\tau_2 > R$.) Suppose we work out the amplitude for this “punched” region. Compatibility then means that: (a) the composition of the amplitude of a punched region with the amplitude of a disk fitting into the hole yields the amplitude of the resulting slice region and (b) the composition of the amplitude of a punched region with the amplitude of an annulus region fitting into the hole yields the amplitude of the resulting punched region.

A direct calculation of the amplitude of a punched region in the way we have done the calculations for the other types of regions would be very complicated and we will not attempt it here. We will, however, be able to infer this

amplitude indirectly. It is a map $\rho_{[\tau_1, \tau_2, R]} : \mathcal{H}_1 \otimes \mathcal{H}_2^* \otimes \mathcal{H}_R^* \rightarrow \mathbb{C}$. Here, \mathcal{H}_1 denotes the Hilbert space associated with the line $\tau = \tau_1$, with its orientation being induced by it being in the boundary of the punched region. \mathcal{H}_2 denotes the Hilbert space associated with the line $\tau = \tau_2$, with the same (translated) orientation, i.e., its orientation is opposite to the one induced by it being in the boundary of the punched region. \mathcal{H}_R is the state space of the circle, oriented as the boundary of a disk. Thinking about the classical boundary value problem we should expect the induced map $\tilde{\rho}_{[\tau_1, \tau_2, R]} : \mathcal{H}_1 \otimes \mathcal{H}_2^* \rightarrow \mathcal{H}_R$ to be an isomorphism. (In the terminology of [2] the state spaces \mathcal{H}_1 and \mathcal{H}_2 are of size $1/2$ while the state space \mathcal{H}_R is of size 1).

The concept of such an isomorphism is familiar from [12, 13], where a similar situation occurs. There, an isomorphism between the tensor product of an initial and final state space (similar to $\mathcal{H}_1 \otimes \mathcal{H}_2^*$ here) and the state space on a hypercylinder (similar to \mathcal{H}_R here) is constructed in Klein-Gordon theory. In that case there is no region that interpolates between the two types of hypersurface and thus no interpolating amplitude.⁴ Instead, the isomorphism emerges from a comparison of amplitudes for the theory coupled to a source. We will follow the same route in the present work.

Nevertheless, we introduce the isomorphism already here since it is valid whether or not we include sources. Only the justification for choosing the isomorphism precisely in this way may seem weak in the present context. We shall see later on, in the context of the theory with source that this choice is forced upon us. An alternative way to justify the exact form of the isomorphism could be to perform a semiclassical analysis (which suggests itself because we are using coherent states). However, we will not do this here.

Let η_1 and η_2 be complex functions on the interval $[-m, m]$ determining coherent states $\psi_{\tau_1, \eta_1} \in \mathcal{H}_1$ and $\overline{\psi_{\tau_2, \eta_2}} \in \mathcal{H}_2^*$. We define the following complex solution of the Helmholtz equation in spacetime,

$$\hat{\eta}(\tau, x) = \int_{-m}^m \frac{d\nu}{2\pi} \frac{1}{2\omega_\nu} \left(\eta_1(\nu) e^{-i\omega_\nu \tau - i\nu x} + \overline{\eta_2(\nu)} e^{i\omega_\nu \tau + i\nu x} \right). \quad (68)$$

This establishes a one-to-one correspondence between bounded complex classical solutions in spacetime and coherent states in $\mathcal{H}_1 \otimes \mathcal{H}_2^*$. Let $\{\xi_n\}_{n \in \mathbb{Z}}$ be complex coefficients defining a coherent state $\psi_{R, \xi} \in \mathcal{H}_R$. We define the following complex solution of the Helmholtz equation in spacetime,

$$\hat{\xi}(r, \vartheta) = \frac{1}{4} \sum_n \xi_n J_n(mr) e^{-in\vartheta}. \quad (69)$$

This establishes a one-to-one correspondence between bounded complex classical solutions in spacetime and coherent states in \mathcal{H}_R .

Now that we have two correspondences between coherent states and classical solutions we use these to identify the coherent states in $\mathcal{H}_1 \otimes \mathcal{H}_2^*$ with those in \mathcal{H}_R . That is, for a complex classical solution $\hat{\zeta}$ in spacetime we compute on the one hand its components ζ_1 and ζ_2 in terms of (68) and on the other hand the coefficients ζ_n in terms of (69). The isomorphism $\mathcal{H}_1 \otimes \mathcal{H}_2^* \rightarrow \mathcal{H}_R$ is then determined by $\psi_{\tau_1, \zeta_1} \otimes \overline{\psi_{\tau_2, \zeta_2}} \mapsto \psi_{R, \zeta}$.

A. Equivalence of amplitudes

We proceed to show that the isomorphism between state spaces does indeed lead to an equivalence between the amplitude for a slice region and the amplitude for a disk region. This amounts to demonstrating the gluing property of the amplitude of the punched region with the amplitude of the disk region. Again, we do not need to fix τ_1 , τ_2 or R since we use the interaction picture. Rather, for convenience, we can think of amplitudes as asymptotic amplitudes, i.e., the expressions (63) and (67). Explicitly, we are going to show that,

$$\mathcal{S}_0(\psi_{\zeta_1} \otimes \overline{\psi_{\zeta_2}}) = \mathcal{S}_0(\psi_{\zeta}). \quad (70)$$

We start by expressing the classical solution $\hat{\zeta}$ in terms of Bessel functions of the first kind. We introduce in (68) the variable $\alpha = \arcsin \frac{\nu}{m}$,

$$\hat{\zeta}(\tau, x) = \int_{-\pi/2}^{\pi/2} \frac{d\alpha}{4\pi} \left(\zeta_1(m \sin \alpha) e^{-im\tau \cos \alpha - imx \sin \alpha} + \overline{\zeta_2(m \sin \alpha)} e^{im\tau \cos \alpha + imx \sin \alpha} \right). \quad (71)$$

⁴ If one introduces “negative” regions then it is possible to consider such amplitudes. But this is another story on which we will not expand here.

In the polar coordinates this expression takes the form

$$\hat{\zeta}(r, \vartheta) = \int_{-\pi/2}^{\pi/2} \frac{d\alpha}{4\pi} \left(\zeta_1(m \sin \alpha) e^{-imr \sin(\alpha+\vartheta)} + \overline{\zeta_2}(m \sin \alpha) e^{imr \sin(\alpha+\vartheta)} \right). \quad (72)$$

We rewrite the exponentials in terms of the Bessel function of the first kind as

$$e^{\pm imr \sin(\alpha+\vartheta)} = \sum_n e^{in(\alpha+\vartheta)} J_n(\pm mr). \quad (73)$$

Hence,

$$\hat{\zeta}(r, \vartheta) = \frac{1}{4} \sum_n J_n(mr) e^{in\vartheta} \int_{-\pi/2}^{\pi/2} \frac{d\alpha}{\pi} e^{in\alpha} (\zeta_1(m \sin \alpha) + (-1)^n \overline{\zeta_2}(m \sin \alpha)). \quad (74)$$

Under the identification of the complex classical solutions of the Helmholtz equation (68) and (69) we obtain,

$$\zeta_{-n} = \int_{-\pi/2}^{\pi/2} \frac{d\alpha}{\pi} e^{in\alpha} (\zeta_1(m \sin \alpha) + (-1)^n \overline{\zeta_2}(m \sin \alpha)). \quad (75)$$

Substituting this in the free amplitude (67),

$$\begin{aligned} \mathcal{S}_0(\psi_\zeta) = & \exp \left(\sum_n \left[(-1)^n \int \frac{d\alpha d\alpha'}{16\pi^2} e^{in(\alpha-\alpha')} (\zeta_1(m \sin \alpha) + (-1)^n \overline{\zeta_2}(m \sin \alpha)) (\zeta_1(m \sin \alpha') + (-1)^{-n} \overline{\zeta_2}(m \sin \alpha')) \right. \right. \\ & \left. \left. - \int \frac{d\alpha d\alpha'}{\pi^2} e^{in(\alpha-\alpha')} (\zeta_1(m \sin \alpha) + (-1)^n \overline{\zeta_2}(m \sin \alpha)) (\overline{\zeta_1}(m \sin \alpha') + (-1)^n \zeta_2(m \sin \alpha')) \right] \right), \end{aligned}$$

Notice that

$$\int_{-\pi/2}^{\pi/2} d\alpha \int_{-\pi/2}^{\pi/2} d\alpha' \delta(\alpha - \alpha' + \pi) = 0. \quad (76)$$

We arrive at

$$\mathcal{S}_0(\psi_\zeta) = \exp \left(\int_{-\pi/2}^{\pi/2} \frac{d\alpha}{4\pi} \left(\zeta_1(m \sin \alpha) \overline{\zeta_2}(m \sin \alpha) - \frac{1}{2} \zeta_1(m \sin \alpha) \overline{\zeta_1}(m \sin \alpha) - \frac{1}{2} \zeta_2(m \sin \alpha) \overline{\zeta_2}(m \sin \alpha) \right) \right). \quad (77)$$

Substituting $\alpha = \arcsin \frac{\nu}{m}$, we obtain

$$\mathcal{S}_0(\psi_\zeta) = \exp \left(\int_{-m}^m \frac{d\nu}{2\pi} \frac{1}{2\omega_\nu} \left[\zeta_1(\nu) \overline{\zeta_2(\nu)} - \frac{1}{2} |\zeta_1(\nu)|^2 - \frac{1}{2} |\zeta_2(\nu)|^2 \right] \right) = \langle \psi_{\zeta_2} | \psi_{\zeta_1} \rangle = \mathcal{S}_0(\psi_{\zeta_1} \otimes \overline{\psi_{\zeta_2}}). \quad (78)$$

This concludes the proof of the equivalence of the free amplitudes (63) and (67).

B. Isomorphism in terms of multiparticle states

We turn to work out the form of isomorphism $\mathcal{H}_1 \otimes \mathcal{H}_2^* \rightarrow \mathcal{H}_R$ in terms of multiparticle states. Note that we are using the interaction picture and hence may omit the labels τ_1 , τ_2 and R . We denote a state in $\mathcal{H}_1 \otimes \mathcal{H}_2^*$ with q incoming particles with quantum numbers ν_1, \dots, ν_q and $k-q$ outgoing particles with quantum numbers ν_{q+1}, \dots, ν_k as

$$\psi_{\nu_1, \dots, \nu_q | \nu_{q+1}, \dots, \nu_k} = |\psi_{\nu_1, \dots, \nu_q}\rangle \otimes \langle \psi_{\nu_{q+1}, \dots, \nu_k}|, \quad (79)$$

where $|\psi_{\nu_1, \dots, \nu_q}\rangle$ is a q -particle state in \mathcal{H}_1 introduced in (47). The scalar product of this k -particle state, $\psi_{\nu_1, \dots, \nu_q | \nu_{q+1}, \dots, \nu_k}$, with a coherent state defined by the complex function $\hat{\zeta}$ results to be

$$\langle \psi_\zeta | \psi_{\nu_1, \dots, \nu_q | \nu_{q+1}, \dots, \nu_k} \rangle = \langle \psi_{\zeta_1} | \psi_{\nu_1, \dots, \nu_q} \rangle \langle \psi_{\nu_{q+1}, \dots, \nu_k} | \psi_{\zeta_2} \rangle = \overline{C_{\zeta_1}} C_{\zeta_2} \zeta_2(\nu_1) \cdots \zeta_2(\nu_q) \overline{\zeta_1}(\nu_{q+1}) \cdots \overline{\zeta_1}(\nu_k), \quad (80)$$

where the relation (48) has been used. With the use of the correspondence (75), we can express this inner product in terms of the modes ζ_n . From (75) we derive the following relations,

$$\zeta_1(\nu) = \sum_n \frac{\zeta_n}{2} e^{in\kappa_\nu}, \quad \overline{\zeta_2}(\nu) = \sum_n (-1)^n \frac{\zeta_n}{2} e^{in\kappa_\nu}, \quad (81)$$

where $\kappa_\nu := \arctan \frac{\nu}{\omega_\nu}$. Inserting these expressions in (80) yields,

$$\langle \psi_{\hat{\zeta}} | \psi_{\nu_1, \dots, \nu_q | \nu_{q+1}, \dots, \nu_k} \rangle = \exp \left(- \sum_n \frac{|\zeta_n|^2}{16} \right) \sum_{n_1, \dots, n_k} (-1)^{n_1 + \dots + n_k} \frac{\overline{\zeta_{n_1}}}{2} \dots \frac{\overline{\zeta_{n_k}}}{2} e^{-i(n_1 \kappa_{\nu_1} + \dots + n_k \kappa_{\nu_k})}. \quad (82)$$

With the identification of the complex classical solutions (68) and (69) and the completeness relation of the coherent states (56), we are now able to express the inner product of a k -particle state in $\mathcal{H}_1 \otimes \mathcal{H}_2^*$ with a k -particle state in \mathcal{H}_R ,

$$\langle \psi_{n_1, \dots, n_k} | \psi_{\nu_1, \dots, \nu_q | \nu_{q+1}, \dots, \nu_k} \rangle = D^{-1} \int d\hat{\zeta} d\bar{\zeta} \langle \psi_{n_1, \dots, n_k} | \psi_{\hat{\zeta}} \rangle \langle \psi_{\hat{\zeta}} | \psi_{\nu_1, \dots, \nu_q | \nu_{q+1}, \dots, \nu_k} \rangle, \quad (83)$$

Inserting (57), (59), (82) in (83) and performing the integration in $d\hat{\zeta} d\bar{\zeta}$, we obtain

$$\langle \psi_{n_1, \dots, n_k} | \psi_{\nu_1, \dots, \nu_q | \nu_{q+1}, \dots, \nu_k} \rangle = (-1)^{n_1 + \dots + n_k} 2^{-k} e^{-i(n_1 \kappa_{\nu_1} + \dots + n_k \kappa_{\nu_k})} \frac{1}{k!} \sum_{\sigma \in S_k} \prod_{i=1}^k \delta_{n_i, n'_{\sigma(i)}}, \quad (84)$$

where the sum runs over all permutations σ of k elements. Hence a k -particle state in $\mathcal{H}_1 \otimes \mathcal{H}_2^*$ can be written as a linear combination of k -particle states in \mathcal{H}_R as,

$$\psi_{\nu_1, \dots, \nu_q | \nu_{q+1}, \dots, \nu_k} = \sum_{n_1, \dots, n_k} (-1)^{n_1 + \dots + n_k} 4^k e^{-i(n_1 \kappa_{\nu_1} + \dots + n_k \kappa_{\nu_k})} \psi_{n_1, \dots, n_k}. \quad (85)$$

Reciprocally, a k -particle state in \mathcal{H}_R is a linear combination of k -particle states in $\mathcal{H}_1 \otimes \mathcal{H}_2^*$,

$$\psi_{n_1, \dots, n_k} = \int \frac{d\nu_1}{2\pi 2\omega_{\nu_1}} \dots \frac{d\nu_k}{2\pi 2\omega_{\nu_k}} (-1)^{n_1 + \dots + n_k} 2^{-k} e^{-i(n_1 \kappa_{\nu_1} + \dots + n_k \kappa_{\nu_k})} \psi_{\nu_1, \dots, \nu_q | \nu_{q+1}, \dots, \nu_k}. \quad (86)$$

VIII. ASYMPTOTIC AMPLITUDES IN THEORY WITH SOURCE

We study in this section the interaction of the field with a source field μ described by the action

$$S_{M,\mu}(\phi) = S_{M,0}(\phi) + \int_M d\tau dx \phi(\tau, x) \mu(\tau, x), \quad (87)$$

where $S_{M,0}$ is the free action (1). The resulting field propagator can be expressed in terms of the one of the free theory. Indeed in the expression of the path integral defining the propagator, we shift the integration variable by a classical solution of the free theory matching the boundary configurations φ on the boundary ∂M , and we obtain the result

$$Z_{M,\mu}(\varphi) = \frac{N_{M,\mu}}{N_{M,0}} Z_{M,0}(\varphi) \exp \left(i \int_M d\tau dx \phi_{cl}(\tau, x) \mu(\tau, x) \right), \quad (88)$$

where the normalization factor $N_{M,\mu}$ is formally equal to

$$N_{M,\mu} = \int_{\phi|_{\partial M}=0} \mathcal{D}\phi e^{iS_{M,\mu}(\phi)}. \quad (89)$$

In order to relate this normalization factor to that of the free theory $N_{M,0}$ (19), we shift the integration variable in (89) by the function α , solution of the inhomogeneous Helmholtz equation

$$(\partial_\tau^2 + \partial_x^2 + m^2) \alpha(\tau, x) = \mu(\tau, x), \quad (90)$$

with the boundary condition $\alpha|_{\partial M} = 0$. We can now rewrite the normalization factor (89) as

$$N_{M,\mu} = N_{M,0} \exp \left(\frac{i}{2} \int_M d\tau dx \mu(\tau, x) \alpha(\tau, x) \right). \quad (91)$$

A. Slice region

We assume that the source field μ vanishes outside the slice region $[\tau_1, \tau_2] \times \mathbb{R}$. We then rewrite the last exponential in (88) using (5) as,

$$\exp\left(i \int d\tau dx \phi_{cl}(\tau, x) \mu(\tau, x)\right) = \exp\left(\int dx (\varphi_1(x) \mu_1(x) + \varphi_2(x) \mu_2(x))\right), \quad (92)$$

where φ_i is the field configuration at τ_i and

$$\mu_1(x) := i \int_{\tau_1}^{\tau_2} d\tau \frac{\sin \omega(\tau_2 - \tau)}{\sin \omega(\tau_2 - \tau_1)} \mu(\tau, x), \quad \mu_2(x) := i \int_{\tau_1}^{\tau_2} d\tau \frac{\sin \omega(\tau - \tau_1)}{\sin \omega(\tau_2 - \tau_1)} \mu(\tau, x). \quad (93)$$

The amplitude $\rho_{[\tau_1, \tau_2], \mu}$ associated with the transition from the coherent state ψ_{τ_1, η_1} at $\tau = \tau_1$ to the coherent state ψ_{τ_2, η_2} at $\tau = \tau_2$ is, in the interaction picture,

$$\begin{aligned} \rho_{[\tau_1, \tau_2], \mu}(\psi_{\tau_1, \eta_1} \otimes \overline{\psi_{\tau_2, \eta_2}}) &= K_{\tau_1, \eta_1} \overline{K_{\tau_2, \eta_2}} \int \mathcal{D}\varphi_1 \mathcal{D}\varphi_2 \exp\left(\int_{-m}^m \frac{d\nu}{2\pi} \left(\eta_1(\nu) e^{-i\omega_\nu \tau_1} \varphi_1(\nu) + \overline{\eta_2(-\nu)} e^{i\omega_\nu \tau_2} \varphi_2(\nu) \right)\right) \\ &\quad \psi_0(\varphi_1) \overline{\psi_0(\varphi_2)} Z_{[\tau_1, \tau_2], \mu}(\varphi_1, \varphi_2). \end{aligned} \quad (94)$$

Introducing two new complex functions $\tilde{\eta}_1$ and $\tilde{\eta}_2$ defined as

$$\tilde{\eta}_1(\nu) := \eta_1(\nu) + \int dx e^{i\omega_\nu \tau_1 + i\nu x} \mu_1(x) \quad \text{and} \quad \tilde{\eta}_2(\nu) := \eta_2(\nu) + \int dx e^{i\omega_\nu \tau_2 + i\nu x} \overline{\mu_2}(x), \quad (95)$$

we can rewrite (94) in the form

$$\rho_{[\tau_1, \tau_2], \mu}(\psi_{\tau_1, \eta_1} \otimes \overline{\psi_{\tau_2, \eta_2}}) = \rho_{[\tau_1, \tau_2], 0}(\psi_{\tau_1, \tilde{\eta}_1} \otimes \overline{\psi_{\tau_2, \tilde{\eta}_2}}) \frac{N_{[\tau_1, \tau_2], \mu} K_{\tau_1, \eta_1} \overline{K_{\tau_2, \eta_2}}}{N_{[\tau_1, \tau_2], 0} K_{\tau_1, \tilde{\eta}_1} \overline{K_{\tau_2, \tilde{\eta}_2}}}. \quad (96)$$

Substituting the expressions of the inner product (62) and the normalization factors (46), we obtain

$$\begin{aligned} \rho_{[\tau_1, \tau_2], \mu}(\psi_{\tau_1, \eta_1} \otimes \overline{\psi_{\tau_2, \eta_2}}) &= \rho_{[\tau_1, \tau_2], 0}(\psi_{\tau_1, \eta_1} \otimes \overline{\psi_{\tau_2, \eta_2}}) \frac{N_{[\tau_1, \tau_2], \mu}}{N_{[\tau_1, \tau_2], 0}} \exp\left(i \int d\tau dx \mu(\tau, x) \hat{\eta}(\tau, x)\right) \cdot \\ &\quad \exp\left(\int \frac{dx}{4\omega} \left(\mu_1^2(x) + \mu_2^2(x) + 2\mu_1(x)e^{-i\omega(\tau_2 - \tau_1)} \mu_2(x) \right)\right), \end{aligned} \quad (97)$$

where the complex function $\hat{\eta}$ is the complex classical solution of the Helmholtz equation determined by the η_1 and η_2 introduced in (68). Substituting the expressions of the function μ_1 and μ_2 given in (93), the last exponential in (97) can be written in the form

$$\exp\left(\int \frac{dx}{4\omega} \left(\mu_1^2(x) + \mu_2^2(x) + 2\mu_1(x)e^{-i\omega(\tau_2 - \tau_1)} \mu_2(x) \right)\right) = \exp\left(\frac{i}{2} \int d\tau dx \mu(\tau, x) \beta(\tau, x)\right), \quad (98)$$

where β is the solution of the Helmholtz equation given by

$$\beta(\tau, x) = \int_{\tau_1}^{\tau_2} \frac{d\tau'}{2\omega} \left(ie^{i\omega(\tau - \tau')} + 2 \frac{\sin(\omega(\tau - \tau_1)) \sin(\omega(\tau_2 - \tau'))}{\sin(\omega(\tau_2 - \tau_1))} \right) \mu(\tau', x). \quad (99)$$

We now turn to the quotient of normalization factors $\frac{N_{[\tau_1, \tau_2], \mu}}{N_{[\tau_1, \tau_2], 0}}$ appearing in (97). It can be expressed using (91) as

$$\frac{N_{[\tau_1, \tau_2], \mu}}{N_{[\tau_1, \tau_2], 0}} = \exp\left(\frac{i}{2} \int d\tau dx \mu(\tau, x) \alpha(\tau, x)\right), \quad (100)$$

where α is a solution of equation (90) with the boundary conditions $\alpha(\tau_1, x) = 0$ and $\alpha(\tau_2, x) = 0$. α results to be

$$\alpha(\tau, x) = \int_{\tau_1}^{\tau_2} \frac{d\tau'}{\omega} \left(\theta(\tau - \tau') \sin(\omega(\tau - \tau')) - \frac{\sin(\omega(\tau - \tau_1)) \sin(\omega(\tau_2 - \tau'))}{\sin(\omega(\tau_2 - \tau_1))} \right) \mu(\tau', x), \quad (101)$$

where $\theta(t)$ is the step function

$$\theta(t) = \begin{cases} 1 & \text{if } t > 0 \\ 0 & \text{if } t < 0. \end{cases} \quad (102)$$

Summing the functions $\alpha(\tau, x)$ given by (101) and $\beta(\tau, x)$ given by (99),

$$\gamma(\tau, x) := \alpha(\tau, x) + \beta(\tau, x) = \int_{\tau_1}^{\tau_2} \frac{d\tau'}{\omega} \left(\theta(\tau - \tau') \sin(\omega(\tau - \tau')) + \frac{i}{2} e^{i\omega(\tau - \tau')} \right) \mu(\tau', x), \quad (103)$$

we obtain a negative frequency solution of the inhomogeneous Helmholtz equation for $\tau < \tau_1$,

$$\gamma(\tau, x) \Big|_{\tau < \tau_1} = \int_{\tau_1}^{\tau_2} \frac{d\tau'}{\omega} \frac{i}{2} e^{i\omega(\tau - \tau')} \mu(\tau', x), \quad (104)$$

and a positive frequency solution for $\tau > \tau_2$,

$$\gamma(\tau, x) \Big|_{\tau > \tau_2} = \int_{\tau_1}^{\tau_2} \frac{d\tau'}{\omega} \frac{i}{2} e^{-i\omega(\tau - \tau')} \mu(\tau', x). \quad (105)$$

We combine the factors (98) and (91) to obtain

$$\exp \left(\frac{i}{2} \int d\tau dx \mu(\tau, x) [\alpha(\tau, x) + \beta(\tau, x)] \right) = \exp \left(\frac{i}{2} \int d\tau dx d\tau' dx' \mu(\tau, x) G(\tau, x, \tau', x') \mu(\tau', x') \right), \quad (106)$$

with

$$\begin{aligned} G(\tau, x, \tau', x') &= i \int_{-\infty}^{\infty} \frac{d\nu}{2\pi} \left[\theta(\tau - \tau') \frac{e^{i\omega_\nu(\tau' - \tau) - i\nu(x - x')}}{2\omega_\nu} + \theta(\tau' - \tau) \frac{e^{-i\omega_\nu(\tau' - \tau) - i\nu(x - x')}}{2\omega_\nu} \right], \\ &= i \int_0^{\infty} \frac{d\nu}{2\pi} \frac{e^{-i\omega_\nu|\tau' - \tau|}}{\omega_\nu} \cos(\nu(x - x')). \end{aligned} \quad (107)$$

We recognize on the right-hand side the integral representation of the Hankel function [16]. Introducing the 2-dimensional vectors \underline{r} and \underline{r}' , with components (τ, x) and (τ', x') respectively, the function $G(\tau, x, \tau', x')$ takes the form

$$G(\underline{r}, \underline{r}') = \frac{i}{4} \overline{H_0}(m|\underline{r} - \underline{r}'|), \quad (108)$$

where H_0 denotes the Hankel function of order 0. Hence the transition amplitude now reads

$$\rho_{[\tau_1, \tau_2], \mu}(\psi_{\tau_1, \eta_1} \otimes \overline{\psi_{\tau_2, \eta_2}}) = \rho_{[\tau_1, \tau_2], 0}(\psi_{\tau_1, \eta_1} \otimes \overline{\psi_{\tau_2, \eta_2}}) \exp \left(i \int d^2x \mu(x) \hat{\eta}(x) \right) \exp \left(\frac{i}{2} \int d^2x d^2x' \mu(x) G(x, x') \mu(x') \right). \quad (109)$$

In order to interpret this transition amplitude as an element of the S-matrix, denoted by S_μ , we take the (trivial) limit $\tau_1 \rightarrow -\infty$ and $\tau_2 \rightarrow +\infty$,

$$\begin{aligned} S_\mu(\psi_{\eta_1} \otimes \overline{\psi_{\eta_2}}) &= \lim_{\substack{\tau_1 \rightarrow -\infty \\ \tau_2 \rightarrow +\infty}} \rho_{[\tau_1, \tau_2], \mu}(\psi_{\tau_1, \eta_1} \otimes \overline{\psi_{\tau_2, \eta_2}}), \\ &= S_0(\psi_{\eta_1} \otimes \overline{\psi_{\eta_2}}) \exp \left(i \int d^2x \mu(x) \hat{\eta}(x) \right) \exp \left(\frac{i}{2} \int d^2x d^2x' \mu(x) G(x, x') \mu(x') \right). \end{aligned} \quad (110)$$

B. Disk region

We now consider a source field μ vanishing outside the disk region. As in the preceding section we start by evaluating the last exponential in (88). With the classical solution (10), the argument of the exponential reads,

$$\int dr d\vartheta r \mu(r, \vartheta) \phi_{cl}(r, \vartheta) = \int dr d\vartheta r \mu(r, \vartheta) \frac{J_n(mr)}{J_n(mR)} \varphi(\vartheta) = \sum_n \varphi_{-n} \frac{2\pi}{J_n(mR)} j_n, \quad (111)$$

with the quantity j_n given by

$$j_n := \int dr r J_n(mr) \mu_n(r). \quad (112)$$

The amplitude of a coherent state in the theory with source is

$$\rho_{R,\mu}(\psi_{R,\xi}) = \frac{N_{R,\mu}}{N_{R,0}} \int \mathcal{D}\varphi \psi_{R,\xi}(\varphi) \exp\left(i \sum_n \varphi_{-n} \frac{2\pi}{J_n(mR)} j_n\right) Z_{R,0}(\varphi). \quad (113)$$

The integration can be performed by introducing a new coherent state defined by the complex function $\tilde{\xi}$ related to ξ via

$$\tilde{\xi}_n := \xi_n + i2\pi \frac{\overline{H}_n(mR)}{J_{-n}(mR)} j_{-n}. \quad (114)$$

Then the amplitude has the form

$$\rho_{R,\mu}(\psi_{R,\xi}) = \frac{N_{R,\mu}}{N_{R,0}} \frac{K_{R,\xi}}{K_{R,\tilde{\xi}}} \rho_{R,0}(\psi_{R,\tilde{\xi}}). \quad (115)$$

The substitution of the expression for the free amplitude (66) of the coherent state defined by $\tilde{\xi}$ and the expression of the normalization factors, (55) gives

$$\rho_{R,\mu}(\psi_{R,\xi}) = \rho_{R,0}(\psi_{R,\xi}) \frac{N_{R,\mu}}{N_{R,0}} \exp\left(\sum_n \left(i\frac{\pi}{2} \xi_n j_n - \frac{\pi^2}{2} j_n \frac{\overline{H}_{-n}(mR)}{J_n(mR)} j_{-n}\right)\right). \quad (116)$$

The first term in the argument of the exponential can be written in position space as

$$\exp\left(i\frac{\pi}{2} \sum_n \xi_n j_n\right) = \exp\left(i \int d^2x \mu(x) \hat{\xi}(x)\right), \quad (117)$$

where the function $\hat{\xi}$ in polar coordinates is given by (69). We express the second term in the exponential of (116) in the form

$$\exp\left(-\frac{\pi^2}{2} \sum_n j_n \frac{\overline{H}_n(mR)}{J_{-n}(mR)} j_{-n}\right) = \exp\left(\frac{i}{2} \int d^2x \mu(x) \beta(x)\right), \quad (118)$$

with β given by its Fourier components

$$\beta_n(r) = i\frac{\pi}{2} J_n(mr) \frac{\overline{H}_n(mR)}{J_n(mR)} j_n. \quad (119)$$

We now consider the quotient $\frac{N_{R,\mu}}{N_{R,0}}$. This factor can be expressed using (91) as,

$$\frac{N_{R,\mu}}{N_{R,0}} = \exp\left(\frac{i}{2} \int d^2x \mu(x) \alpha(x)\right), \quad (120)$$

where the function α now satisfies the inhomogeneous Helmholtz equation in polar coordinates (6) with the boundary condition $\alpha(R, \vartheta) = 0$. It will be convenient to work in momentum space: We consider the Fourier components of α and μ ,

$$\alpha(r, \vartheta) = \sum_n \alpha_n(r) e^{in\vartheta}, \quad \mu(r, \vartheta) = \sum_n \mu_n(r) e^{in\vartheta}. \quad (121)$$

The inhomogeneous Helmholtz equation takes the form

$$\left(\partial_r^2 + \frac{1}{r} \partial_r + \left(m^2 - \frac{n^2}{r^2}\right)\right) \alpha_n(r) = \mu_n(r). \quad (122)$$

The solution is

$$\alpha_n(r) = i\frac{\pi}{2} \left(J_n(mr) \left[h_n(r) - h_n + \frac{H_n(mR)}{J_n(mR)} j_n \right] - H_n(mr) j_n(r) \right), \quad (123)$$

where

$$h_n(r) := \int_0^r ds s H_n(ms) \mu_n(s), \quad (124)$$

$$h_n := \int_0^\infty ds s H_n(ms) \mu_n(s), \quad (125)$$

$$j_n(r) := \int_0^r ds s J_n(ms) \mu_n(s). \quad (126)$$

Summing the functions $\alpha_n(r)$ given by (123) and $\beta_n(r)$ given by (119),

$$\gamma_n(r) := \alpha_n(r) + \beta_n(r) = i\frac{\pi}{2} (J_n(mr) [h_n(r) - h_n + 2j_n] - H_n(mr) j_n(r)), \quad (127)$$

we obtain a solution of the inhomogeneous Helmholtz equation with the following behavior outside the disk region,

$$\gamma_n(r)|_{r>R} = i\frac{\pi}{2} \overline{H}_n(mr) j_n. \quad (128)$$

So, combining the factor (120) and (118) we arrive at

$$\exp \left(\frac{i}{2} \int d^2x \mu(x) [\alpha(x) + \beta(x)] \right) = \exp \left(\frac{i}{2} \int d^2x d^2x' \mu(x) G(x, x') \mu(x') \right), \quad (129)$$

where G is the Green function given in polar coordinates by

$$G(r, \vartheta, r', \vartheta') = -\frac{i}{4} \sum_n e^{in(\vartheta-\vartheta')} (\theta(r-r') J_n(mr') H_n(mr) + \theta(r'-r) J_n(mr) H_n(mr') - 2 J_n(mr') J_n(mr)). \quad (130)$$

Using the addition theorems of the Bessel functions (11.3.4) and (11.3.5) of [17], we can perform the sum over n in (130); we obtain

$$G(r, \vartheta, r', \vartheta') = \frac{i}{4} \overline{H}_0(m\sqrt{r^2 + r'^2 - 2rr' \cos(\vartheta - \vartheta')}). \quad (131)$$

If we note by \underline{r} the vector with polar coordinates (r, ϑ) , the Green function can be written as in equation (108). The amplitude of a coherent state in presence of a source is then

$$\rho_{R,\mu}(\psi_{R,\xi}) = \rho_{R,0}(\psi_{R,\xi}) \exp \left(i \int d^2x \mu(x) \hat{\xi}(x) \right) \exp \left(\frac{i}{2} \int d^2x d^2x' \mu(x) G(x, x') \mu(x') \right). \quad (132)$$

This expression is independent of the radius R (since we are working in the interaction picture). The asymptotic amplitude, \mathcal{S}_μ , is then immediately obtained,

$$\mathcal{S}_\mu(\psi_\xi) = \lim_{R \rightarrow \infty} \rho_{R,\mu}(\psi_{R,\xi}) = \mathcal{S}_0(\psi_\xi) \exp \left(i \int d^2x \mu(x) \hat{\xi}(x) \right) \exp \left(\frac{i}{2} \int d^2x d^2x' \mu(x) G(x, x') \mu(x') \right). \quad (133)$$

C. Comparison of amplitudes with source

Recall from Section VII that there is an isomorphism between the state space $\mathcal{H}_1 \otimes \mathcal{H}_2^*$ on the boundary of a slice region $[\tau_1, \tau_2] \times \mathbb{R}$ and the state space \mathcal{H}_R on the boundary of a disk region S_R^2 in the free theory. Moreover, under this isomorphism the amplitude of the slice region and the disk region become equal. We can extend such a comparison now to amplitudes for the theory coupled to a source. To this end we consider the asymptotic amplitudes for $\tau_1 \rightarrow -\infty$, $\tau_2 \rightarrow +\infty$ and $R \rightarrow \infty$. (Alternatively, we could compare amplitudes for finite time intervals and radii as long as the source is confined completely inside the regions under consideration.) Indeed, comparing (110) with (133) we observe that the two expressions become equal under the isomorphism of Section VII: The first factor in both expressions is the free amplitude which was already shown to be equal. The second factor in both expressions involves complex classical solutions of the Helmholtz equation. It were precisely these complex classical solutions that we used to define the isomorphism. The third factor is obviously equal in both expressions.

In Section VII the choice of the isomorphism $\mathcal{H}_1 \otimes \mathcal{H}_2^* \rightarrow \mathcal{H}_R$ seemed somewhat ad hoc. At this point it becomes clear that there is no other choice. In order for the amplitudes (110) and (133) to be equal we need precisely that the complex classical solutions $\hat{\eta}$ and $\hat{\xi}$ that appear in the expressions for these amplitudes be equal.

IX. GENERAL INTERACTIONS

To describe general perturbative interactions we use the usual technique of functional derivatives with respect to the source field. Thus consider the action

$$S_{M,V}(\phi) = S_{M,0}(\phi) + \int_M d^2x V(x, \phi(x)), \quad (134)$$

where $S_{M,0}$ is the free action (1) and V a potential. We notice the usual functional identity,

$$\exp(iS_{M,V}(\phi)) = \exp\left(i \int_M dx^2 V\left(x, -i \frac{\partial}{\partial \mu(x)}\right)\right) \exp(iS_{M,\mu}(\phi)) \Big|_{\mu=0}, \quad (135)$$

where $S_{M,\mu}$ is the action in the presence of a source interaction, defined in (87). At first we assume the source to vanish outside the region M . We then notice that we can perform all the calculations of Section VIII with the action (134) by always pulling out to the left the factor in (135) with the functional derivative. This finally leads to the interacting S-matrix in functional form. For the asymptotic slice regions this is,

$$\mathcal{S}_V(\psi_1 \otimes \overline{\psi_2}) = \exp\left(i \int d^2x V\left(x, -i \frac{\partial}{\partial \mu(x)}\right)\right) \mathcal{S}_\mu(\psi_1 \otimes \overline{\psi_2}) \Big|_{\mu=0}, \quad (136)$$

while for the asymptotic disk region this is,

$$\mathcal{S}_V(\psi) = \exp\left(i \int d^2x V\left(x, -i \frac{\partial}{\partial \mu(x)}\right)\right) \mathcal{S}_\mu(\psi) \Big|_{\mu=0}. \quad (137)$$

X. CONCLUSIONS

We have presented the Riemannian quantum theory of a field obeying Helmholtz's equation of motion in two-dimensional Euclidean spacetime within the general boundary formulation in two different settings. The first setting is conceptually identical to what is usually done in standard QFT: The state space is defined on a hypersurface of constant time and the evolution of states is considered from one such hypersurface to another. On the other hand, the second setting we studied is incompatible with standard QFT methods of describing the dynamics of quantized fields. The novelty here consists of dealing with a compact spacetime region, the disk region, bounded by one closed line, the circle. We have shown that this second way to describe the quantum theory is completely compatible with the first one: The physical predictions of the two treatments are indeed the same due to the equivalence of the asymptotic amplitudes both for the free and the general interacting theory. This equivalence relies on the existence of an isomorphism between the state spaces defined in the two settings.

Transition amplitudes in the “traditional” setting of slice regions are unitary as should be expected.⁵ Less conventionally, radial “translation” amplitudes between circles in the disk/annulus region setting are also unitary. This means that quantum mechanically probabilities are conserved under “radial evolution”.⁶ Indeed, this should be expected from the classical field theory. Solutions of the Helmholtz equation in a finite region have a unique continuation beyond that region. Thus, “radial evolution” is well defined classically. The field dynamics in a smaller and a larger disk are entirely equivalent. We have thus shown that this is also true quantum mechanically. Note that this is analogous to what happens in Klein-Gordon theory for hypercylinders in Minkowski space [11, 12, 13].

The relevance of the result presented here is that the formulation of the theory in the disk region implements a fully local description of the quantum dynamics of the field. Moreover, one can view this as a kind of finite spacetime holography: The dynamics in a finite spacetime region is completely described through states on the region’s boundary. Any physical interaction between the region and its spacetime surroundings factors through the boundary state space.

An important next step will be the realization of general boundary amplitudes and state spaces for finite regions in a Lorentzian quantum field theory in Minkowski space. While we expect such a description to be feasible, it involves technical challenges related to the fact that the boundary of such a region would have spacelike as well as timelike parts. Furthermore, the solutions of classical field equations in finite spacetime regions no longer determine

⁵ We emphasize again that we are working in a Riemannian real-time setting and not in a Wick rotated setting.

⁶ See [2] for the appropriately generalized notion of probability conservation applicable here.

unique continuations outside such regions. We expect this classical fact to be reflected in the quantum theory in that amplitudes corresponding to annulus like regions no longer permit a representation as unitary operators between the inner and outer boundary state spaces.

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